An Approximation of Minimax Control using Random Sampling and Symbolic Computation

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Outline

Motivation

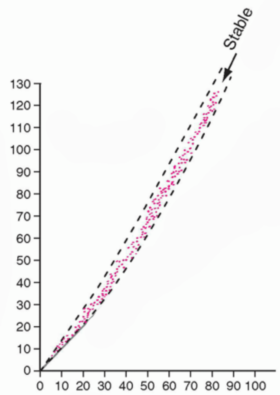
- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

- synchronization
- behavior
- robust control

Robustness

- A control is considered robust if the dynamical system still stable, which means small perturbations to the solution using this control lead to a new solution that stays close to the original solution forever.
- A stable system produces a bounded output for a given bounded input.



Description of the method

- We consider a dynamic system with control $u(\cdot)$ and a bounded disturbance function $d(\cdot)$ over a domain D). The control $u(\cdot)$ is a piecewise constant function that changes of value only at times $t = \tau, 2\tau, \ldots$.
- The set of possible controls of the system for $t \in [0, T]$ is finite and can be described by the set $\mathcal{U} \equiv U^K$ where $T = K\tau$ and $K \in \mathbb{N}$.

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- We suppose given a cost function $C : \mathbb{R}^n \times U^K \to \mathbb{R}_{\geq 0}$, which allows calculating the value $\int_0^T C(x(t), u(t))dt$ for any solution $x(t)$ of $\dot{x}(t) = f(x(t), u(t), d(t))$ for $t \in [0, T]$.

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- The *minimax method* aims to find a control v defined by:

$$v = \arg\min_{u \in U^K} \max_{x(\cdot)} \mathcal{J}_{z_0,\varepsilon}(x(\cdot), u(\cdot))$$

with $\mathcal{J}_{z_0,\varepsilon}(x(\cdot), u(\cdot)) \equiv \left\{ \int_0^T C(x(t), u(t))dt \mid \exists d(\cdot) \in D : \dot{x}(t) = f(x(t), u(t), d(t)) \text{ for } t \in [0, T] \wedge x(0) \in B(z_0, \varepsilon) \right\}$.

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- We propose here a simplified method composed of two steps.

Description of the method

First Step:

- Obtain an upper-bound $\mathcal{K}_{z_0,\varepsilon}(u(\cdot))$ of $\max_{x(\cdot)} \mathcal{J}_{z_0,\varepsilon}(x(\cdot), u(\cdot))$ using an Euler-based symbolic computation method, with:
  $\mathcal{K}_{z_0,\varepsilon}(u(\cdot)) \equiv \max_{x(\cdot)\in B(\tilde{x}^{u(\cdot)}_{z_0}(\cdot),\delta^{u(\cdot)}_{\varepsilon,D}(\cdot))} \{\int_0^T C(x(t), u(t))dt\}$,
  where
  - $\tilde{x}^{u}_{z_0}(\cdot)$ denotes Euler's approximate solution of $\dot{x}(t) = f(x(t), u(t), \mathbf{0})$ for $t \in [0, T]$ with *null perturbation* (i. e., $d(\cdot) = 0$) and initial condition $z_0 \in \mathbb{R}^n$,
  - $\delta^{u(\cdot)}_{\varepsilon,D}(\cdot)$ denotes the upper-bound of the distance between an exact solution and an Euler approximate solution,
  - $x(\cdot) \in B(\tilde{x}^{u(\cdot)}_{z_0}(\cdot), \delta^{u(\cdot)}_{\varepsilon,D}(\cdot))$ means, for all $t \in [0, T]$: $x(t) \in B(\tilde{x}^{u(\cdot)}_{z_0}(t), \delta^{u(\cdot)}_{\varepsilon,D}(t))$. In particular $x(0) \in B(z_0, \varepsilon)$.[1]

[1] $y \in B(z, a)$ with $y, z \in \mathbb{R}^n$ and $a \geq 0$ means $\|y - z\| \leq a$ where $\|\cdot\|$ denotes the Euclidean norm.

Description of the method

Second Step:

- We will not consider the absolute minimum, but a *probable near-minimum* of $\mathcal{K}_{z_0,\varepsilon}(u(\cdot))$ (see [Vid01]).
- The probably approximate near-minimum of $\mathcal{K}_{z_0,\varepsilon}$ is obtained by drawing randomly N control $u_1, \cdots, u_N$ of U^K, i. e., by generating N independent identically distributed (i.i.d.) samples $u_1, \cdots, u_N$ of U^K, with a uniform probability (i. e., with probability $1/|U|^N$) then by taking $\mathcal{K}_{z_0,\varepsilon}(u_N^*)$ with $u_N^* = \arg\min_{u_1,\cdots,u_N} \mathcal{K}_{z_0,\varepsilon}(u_i)$.

Description of the method

Advantage of the method:

- Avoid the excessive complexity of minimax methods,
- Use of samples with large size, as is often the case in statistical learning.
- Take into account *constraints* on the state of the system during its evolution.

Biochemical process example

Consider a biochemical process model Y of continuous culture fermentation (see [**HouskaCDC09**]) and initial condition in $B_0 = B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$ (see [BQ20]):. Let $Y = (X, S, P) \in \mathbb{R}^3$ satisfies the differential system:

$$\begin{cases} \frac{dX}{dt} = -DX(t) + \mu(t)X(t) \\ \frac{dS}{dt} = D\big(S_f(t) - S(t)\big) - \frac{\mu(t)X(t)}{Y_{x/s}} \\ \frac{dP}{dt} = -DP + \big(\alpha\mu(t) + \beta\big)X(t) \end{cases}$$

[BQ20] J. B. van den Berg and E. Queirolo, "A general framework for validated continuation of periodic orbits in systems of polynomial ODEs," **Journal of Computational Dynamics**, vol. 0, no. 2158-2491-2019-0-10, 2020, ISSN: 2158-2491. DOI: 10.3934/jcd.2021004.

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The model is controlled by $S_f \in [S_f^{min}, S_f^{max}]$ and the specific growth rate $\mu : \mathbb{R} \to \mathbb{R}$ of the biomass is a function of the states:

$$\mu(t) = \mu_m \frac{\left(1 - \frac{P(t)}{P_m}\right) S(t)}{K_m + S(t) + \frac{S(t)^2}{K_i}}$$

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Biochemical process example

- Maximize the average productivity presented by the cost function:

$$\mathcal{J}_{z_0,\varepsilon}(x(\cdot), S_f(\cdot)) = \frac{1}{T}\int_0^T DP(t)dt$$

- While satisfying the constraint on the state X:

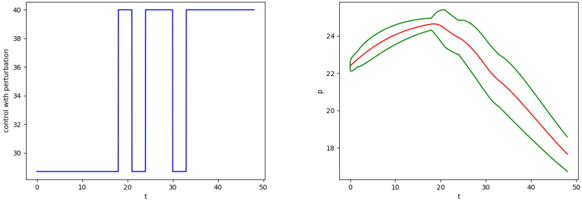
$$\frac{1}{T}\int_0^T X(t)dt \leq 5.8$$

Biochemical process example

- The set $\mathcal{S} \subset \mathbb{R}^n \equiv [3, 8] \times [10, 28] \times [15.5, 25.5]$.
- The codomain $[28.7, 40]$ of the original continuous control function $S_f(\cdot)$ is discretized into a finite set U. After discretization, $S_f(\cdot)$ is a piecewise-constant function that takes its values in the finite set U made of 2 values uniformly taken in $\{28.7, 40\}$.
- We take: $z_0 = (6.52, 12.5, 22.40)$, $\tau = 3$, $\Delta t = \tau/100^2$, $T = 48$, $K = T/\tau = 16$. We consider an additive disturbance d with $d(\cdot) \in \mathcal{D} = [-0.05, 0.05]$.
- In total, we have $2^k = 2^{16}$ possible control cases.

[2] Δt is the "sub-sampling' parameter of the Euler scheme.

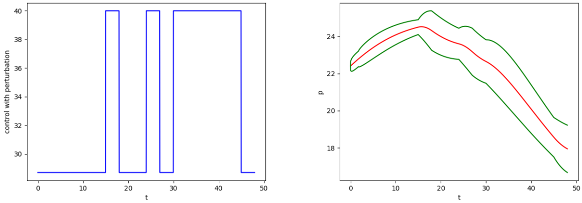
Biochemical process example



Left: control u^* satisfying the constraint on X, obtained by selection among 655 samples picked randomly; right: $P(t)$ under u^* without perturbation (red curve) and with an additive perturbation $d \in [-0.05, 0.05]$ (green curve) over 1 period ($T = 48$) for $\Delta t = 1/400$ and initial condition $(X(0), S(0), P(0)) = (6.52, 12.5, 22.4)$.

- We randomly pick one sample over every 100 possible controls, which gives $2^{16}/100 \approx 655$ samples.
- We get: $\mathcal{K}_{z_0,\varepsilon}(u^*) = 3.1618$ (the constraint on the state X is satisfied since $\frac{1}{T}\int_0^T X(t)dt = 5.782 \leq 5.8$). The CPU computation time of this example is 7 seconds.

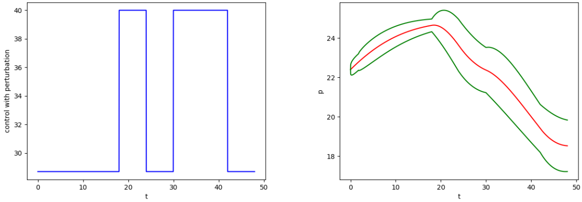
Biochemical process example



Left: control u^* satisfying the constraint on X, obtained by selection among 6554 samples picked randomly; right: $P(t)$ under u^* without perturbation (red curve) and with an additive perturbation $d \in [-0.05, 0.05]$ (green curve) over 1 period ($T = 48$) for $\Delta t = 1/400$ and initial condition $(X(0), S(0), P(0)) = (6.52, 12.5, 22.4)$.

- We randomly pick one sample over every 10 possible controls, which gives $2^{16}/10 \approx 6,554$ samples.
- We get: $\mathcal{K}_{z_0,\varepsilon}(u^*) = 3.1667$. (the constraint on the state X is satisfied since $\frac{1}{T}\int_0^T X(t)dt = 5.794 \leq 5.8$) The CPU computation time of this example is 18.69 seconds.

Biochemical process example



Left: control u^* satisfying the constraint on X, obtained by selection among 65536 samples picked randomly; right: $P(t)$ under u^* without perturbation (red curve) and with an additive perturbation $d \in [-0.05, 0.05]$ (green curve) over 1 period ($T = 48$) for $\Delta t = 1/400$ and initial condition $(X(0), S(0), P(0)) = (6.52, 12.5, 22.4)$.

- We consider all the possible controls, which gives $2^{16} = 65,536$ samples. (The computation is tractable in this example because the set U contains only 2 modes, and because the length K of the horizon is moderate.)
- We get: $\mathcal{K}_{z_0,\varepsilon}(u^*) = 3.1677$ (the constraint is satisfied since $\frac{1}{T}\int_0^T X(t)dt = 5.7995 \leq 5.8$). The CPU computation time of this example is 200 seconds.

Conclusion

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- We showed that the simple combination of random sampling with a symbolic computation method allows to deal with robust optimization problems for nonlinear systems on non-convex domains.
- The method doesn't contain sophisticated theories such as analysis of viscosity solutions of the Hamilton-Jacobi-Bellman-Isaacs equation.

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[Vid01] M. Vidyasagar, "Randomized algorithms for robust controller synthesis using statistical learning theory," **Automatica**, vol. 37, no. 10, pp. 1515–1528, 2001, ISSN: 0005-1098. DOI: 10.1016/S0005-1098(01)00122-4.