Guaranteed phase synchronization of hybrid oscillators using symbolic Euler's method

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Outline

Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

- stability compared to the initial conditions
- behavior
- synchronization

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⇒ The importance of studying:

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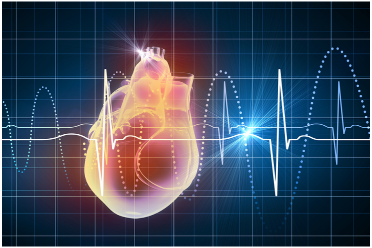
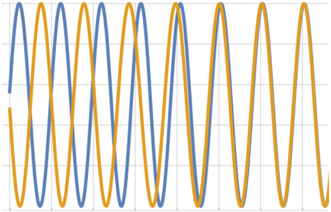


Heart rate variability (HRV)

Solar System

Synchronization

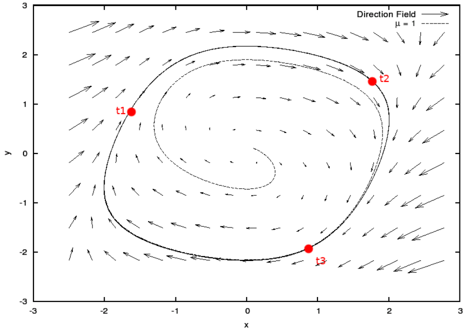
- Adjustment of rhythms of active oscillatory objects due to their weak interaction
- Coordination of multiple events.

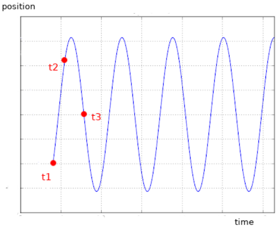


Two oscillators in phase after a lapse of time

How to highlight the synchronization of dynamical system formally?

- Challenge of describing such systems because their equations are non-linear.
- To study non-linear systems, we often visualize them in a space of configurations (position and speed).





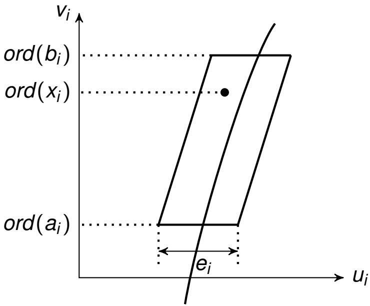
Synchronization using a reachability method

We consider a system composed of 2 subsystems governed by a system of differential equations (ODEs) of the form $\dot{x}(t) = f(x(t))$. The system of ODEs is thus of the form:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t)) \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t)) \end{cases} \tag{1}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$, where m is the dimension of the state space of each subsystem.

Synchronization using a reachability method

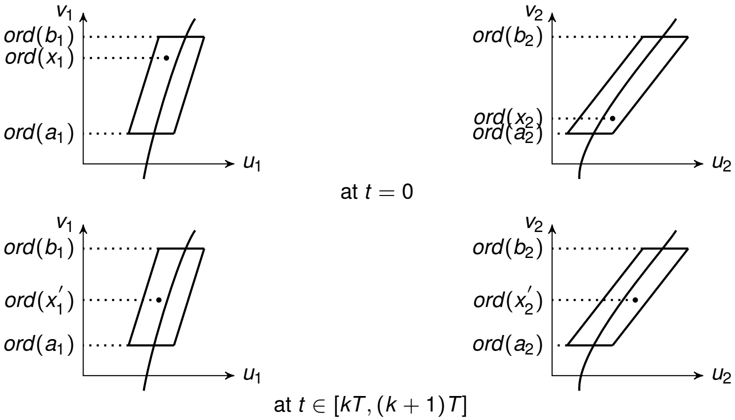


Given a point of $x_i(t)$ of $S_i \equiv (a_i, b_i, e_i)$ at time t, we can define its *phase* $\phi[x_i(t)]$by:

$$\phi[x_i(t)] = (ord(x_i(t)) - ord(a_i))/(ord(b_i) - ord(a_i))$$

where a_i, b_i are the end points of its main diagonal, e_i the size of its horizontal base and $ord(a_i)$ (resp. $ord(b_i)$) denotes the ordinate of a_i (resp. b_i).

Synchronization using a reachability method



Scheme of S_1 (left) and S_2 (right) at $t = 0$ (top) and for some $t \in [kT, (k+1)T)$ (bottom)

Symbolic reachability using Euler's method

- We use the *symbolic Euler's method* [LCDVCF17,Fri17]
- We consider a subset $B = B_1 \times B_2$, where $B_i \subset \mathbb{R}^m$ ($i = 1, 2$) is a ball of the form $\mathcal{B}(c_i, r)$ with c_i is the *centre* and r is the *radius*.

In order to compute the set of solutions starting at B^0. We define for $t \geq 0$:

$$B^{euler}(t) = \mathcal{B}(c_1(t), r(t)) \times \mathcal{B}(c_2(t), r(t)),$$

where $(c_1(t), c_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$ is the approximated value of solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x(0) = (c_1^0, c_2^0)$ given by *Euler's explicit method*, and $r(t) \approx r^0 e^{\lambda t}$ is the *expanded* radius using the *one-sided Lipschitz constant* λ [Söd06].

[LCDVCF17] A. Le Coënt **et al.**, "Control synthesis of nonlinear sampled switched systems using Euler's method," in **SNR**, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.

[Fri17] L. Fribourg, "Euler's method applied to the control of switched systems," in **FORMATS**, (Sep. 5, 2017–Sep. 7, 2017), ser. LNCS, vol. 10419, Berlin, Germany: Springer, Sep. 2017, pp. 3–21. DOI: 10.1007/978-3-319-65765-3_1. [Online]. Available: https://doi.org/10.1007/978-3-319-65765-3_1.

[Söd06] G. Söderlind, "The logarithmic norm. History and modern theory," **BIT Numerical Mathematics**, vol. 46, no. 3, pp. 631–652, 2006, ISSN: 1572-9125. DOI: 10.1007/s10543-006-0069-9. [Online]. Available: https://doi.org/10.1007/s10543-006-0069-9.

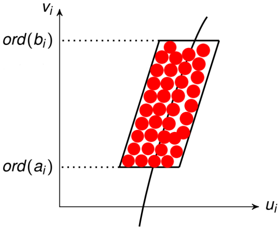
Symbolic reachability using Euler's method

Proposition

Given a covering $\{B_j\}_{j \in J_i}$ *of* S_i *(*$i = 1, 2$*).*
If for all $(j_1, j_2) \in J_1 \times J_2$*,* $PROC1(B_{j_1} \times B_{j_2})$ *succeeds.*
Then, for all initial condition $(x_1^0, x_2^0) \in S$*, there exists* $t \in [kT, (k+1)T)$ *such that* $(x_1(t), x_2(t)) \in S$*. Besides:*

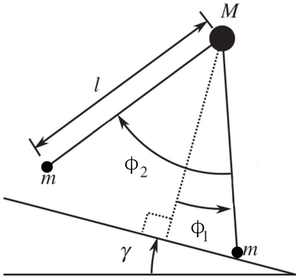
$$|phase(x_1(t)) - phase(x_2(t))| \leq \epsilon + \min(e_1/f_1, e_2/f_2)$$

where $f_i = |ord(b_i) - ord(a_i)|$*.*



Biped

The passive biped model [McG90], seen as a hybrid oscillator, exhibits indeed a stable limit-cycle oscillation for appropriate parameter values that corresponds to periodic movements of the legs [SKN17].



Biped walker

[McG90] T. McGeer, "Passive dynamic walking," **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.

[SKN17] S. Shirasaka, W. Kurebayashi, and H. Nakao, "Phase reduction theory for hybrid nonlinear oscillators," **Physical Review E**, vol. 95, 1 Jan. 2017. DOI: 10.1103/PhysRevE.95.012212.

Biped example

The model has a continuous state variable $\boldsymbol{x}(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))^\top$. The dynamics is described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with:

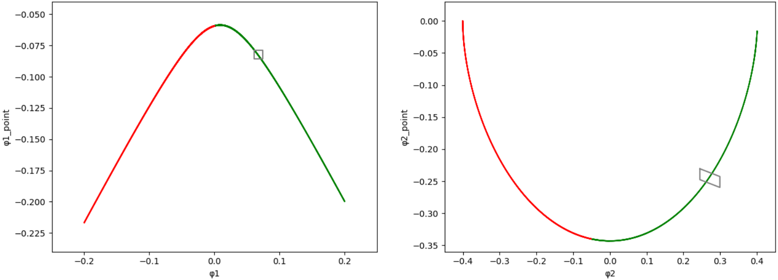
$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \dot{\phi}_1 \\ \sin(\phi_1 - \gamma) \\ \dot{\phi}_2 \\ \sin(\phi_1 - \gamma) + \dot{\phi}_1^2 \sin\phi_2 - \cos(\phi_1 - \gamma)\sin\phi_2 \end{pmatrix} \quad (2)$$

$$Reset(\boldsymbol{x}) = \begin{pmatrix} -\phi_1 \\ \dot{\phi}_1 \sin(2\phi_1) \\ -2\phi_1 \\ \dot{\phi}_1 \cos 2\phi_1 (1 - \cos 2\phi_1) \end{pmatrix} \quad (3)$$

$$Guard(\boldsymbol{x}) \equiv (2\phi_1 - \phi_2 = 0 \ \wedge \phi_2 < -\delta). \quad (4)$$

with $\delta = 0.1$ and $\gamma = 0.009$.

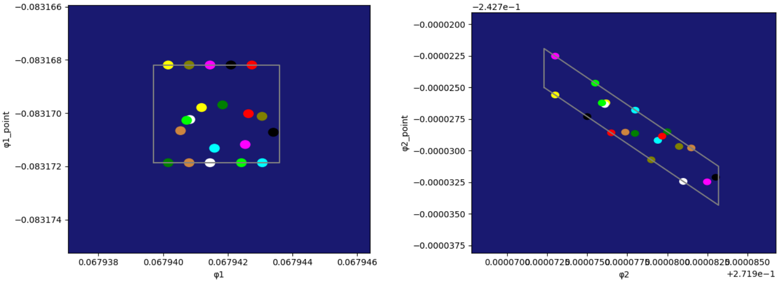
Biped example



Biped: A cyclic trajectory for plan ϕ_1 (left) and ϕ_2 (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

- The time-step used in Euler's method is $\tau = 2 \cdot 10^{-5}$.
- The period of the system is $T = 776440\tau$.
- The radius expansion factor after one period is $E = 2.63$.
- The number of periods considered for synchronization is $k = 30$.

Biped example



Biped: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 30$ periods (without radius expansion for clarity).

Conclusion and Perspectives

Conclusion

- We presented a symbolic reachability method to prove phase synchronization of oscillators.
- A *finite* number of points, displaced from their original position on a synchronization orbit, return after some time into a close neighborhood of the orbit.

Perspectives

- Adapt the *phase reduction* to solve systems with higher state space dimension.
- Replace the symbolic Euler's method by any other symbolic reachability procedure to cover larger sets S.

L. Fribourg, "Euler's method applied to the control of switched systems," in **FORMATS**, (Sep. 5, 2017–Sep. 7, 2017), ser. LNCS, vol. 10419, Berlin, Germany: Springer, Sep. 2017, pp. 3–21. DOI: 10.1007/978-3-319-65765-3_1. [Online]. Available: https://doi.org/10.1007/978-3-319-65765-3_1.

A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg, "Control synthesis of nonlinear sampled switched systems using Euler's method," in **SNR**, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.

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